

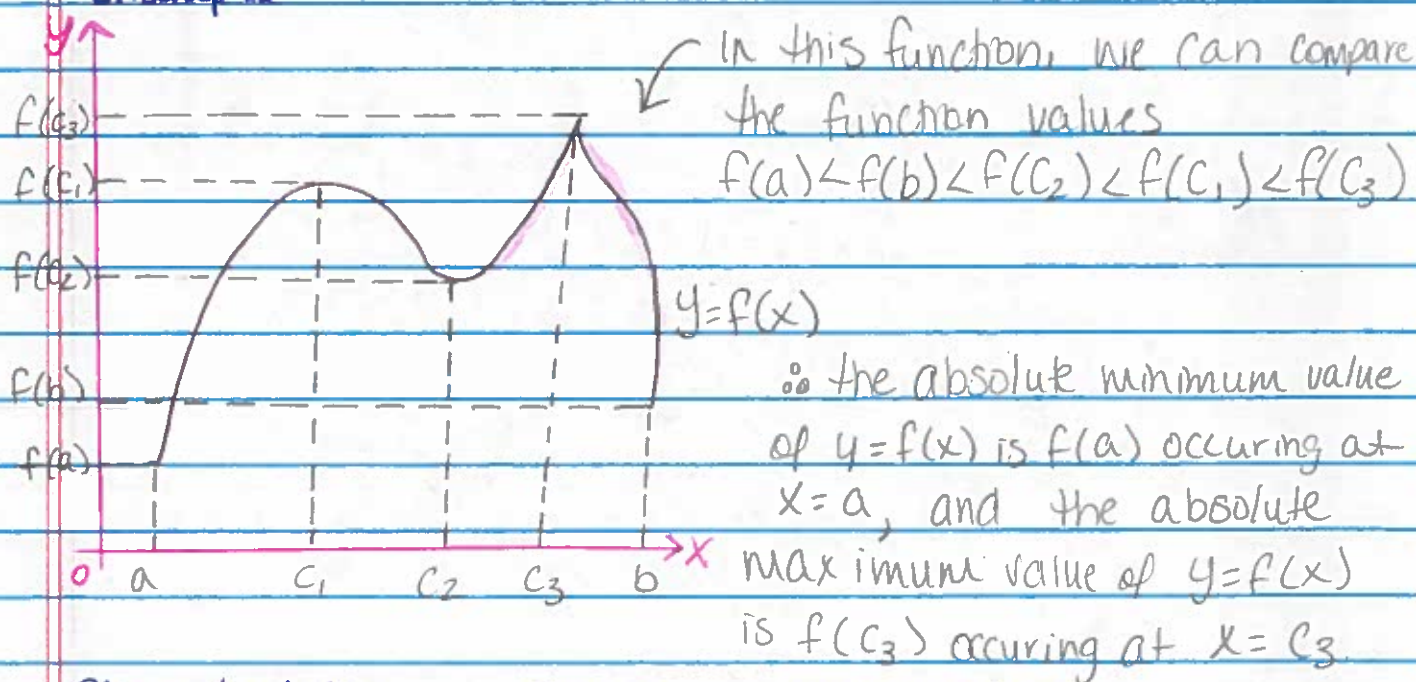
Absolute Extreme values

* A closed interval means on a closed interval $a \leq x \leq b$.

the end points are included. (i) The absolute maximum value of $y = f(x)$ on $a \leq x \leq b$ occurs at $x = c$, where $a \leq c \leq b$, if $f(x) \leq f(c)$ for all x on that interval.

* An open interval means the end points are not included. (ii) The absolute minimum value of $y = f(x)$ on $a \leq x \leq b$ occurs at $x = d$, where $a \leq d \leq b$, if $f(x) \geq f(d)$ for all x on that interval.

Example



Steps to determine absolute extreme values

(i) Find the critical points. Accept points on interval $a \leq x \leq b$; reject points outside of this interval.

(ii) Find the function values at the critical points and end points.

(iii) Compare function values. The smallest function value is the absolute minimum, and the highest is the absolute maximum.

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Extreme value theorem: Every continuous function on a closed interval $[a, b]$ takes on at least one absolute maximum and at least one absolute minimum on that interval.

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* closed function →

The variable cost v is known to be related to the quantity of tires produced q by the quadratic function $V = 0.0002q^2 - 0.06q + 25$, $[0, 280]$.

Determine the output level of production that minimizes the variable cost.

Find the critical points using the first derivative.

① Given $V = 0.0002q^2 - 0.06q + 25$

$$\Rightarrow \frac{dv}{dx} = 0.0004q - 0.06$$

critical points occur when i) $\frac{dv}{dx} = 0$

$$\Rightarrow 0.0004q - 0.06 = 0$$

$$\Rightarrow q = \frac{0.06}{0.0004} = \frac{600}{4} = 150 \leftarrow \text{we accept this critical point since it is on the interval between } [0, 280]$$

ii) $\frac{dv}{dx}$ DNE. ; $\frac{dv}{dq} = 0.0004q - 0.06$ exists everywhere, so this criteria does not yield any critical point.

② calculate function values at the critical point + endpoints

q	$V(q)$
0	\$25 $\leftarrow V(0) = 0.0002(0^2) - 0.06(0) + 25$
150	\$20.50 $\leftarrow V(150) = 0.0002(150^2) - 0.06(150) + 25$
280	\$23.88 $\leftarrow V(280) = 0.0002(280^2) - 0.06(280) + 25$

③ compare the function values

The smallest variable cost is \$20.50 if $q = 150$ tires are produced.

If you spend x amount of money on advertising, then your total cost becomes

$$C = \text{Fixed costs} \\ + (\text{variable cost per unit})(\text{Number of units}) \\ + \text{advertising costs}$$

$$\Rightarrow C = F + vq + x$$

Where F = fixed costs

v = variable costs per unit

q = number of units produced

and x = advertising costs

|| P150

given $P = \$500$, $F = \$30$, $V = \$10$

Suppose x = advertising costs, this yields $q = 0.2\sqrt{x}$

Domain = $\$0 \leq x \leq \4000

i) $R = Pq$

$$\Rightarrow R = (500)(0.2\sqrt{x})$$

$$\Rightarrow R(x) = 100\sqrt{x}; \quad 0 \leq x \leq 4000$$

ii) $C = F + vq + x$

$$C = 30 + (10)(0.2\sqrt{x}) + x$$

$$\Rightarrow C(x) = 30 + 2\sqrt{x} + x; \quad 0 \leq x \leq 4000$$

iii) $P = R - C$

$$= 100\sqrt{x} - (30 + 2\sqrt{x} + x)$$

$$= 100\sqrt{x} - 30 - 2\sqrt{x} - x$$

$$\Rightarrow P(x) = 98\sqrt{x} - 30 - x, \quad 0 \leq x \leq 4000$$

Now, we find the absolute maximum profit. We can see that $P(x)$ is continuous on the closed

Interval so we proceed.

- ① $\hookrightarrow$ Critical points

$$\frac{dP}{dx} = 98\left(\frac{1}{2\sqrt{x}}\right) - 1$$

$$\Rightarrow \frac{dP}{dx} = \frac{49}{\sqrt{x}} - 1$$

$$= \frac{49}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow \frac{dP}{dx} = \frac{49 - \sqrt{x}}{\sqrt{x}}$$

Critical points occur when...

$$i) \frac{dP}{dx} = 0$$

$$\Rightarrow \frac{49 - \sqrt{x}}{\sqrt{x}} = 0$$

$$\Rightarrow 49 - \sqrt{x} = 0$$

$$\Rightarrow \sqrt{x} = 49$$

$$\Rightarrow x = 49^2 = 2401 \leftarrow \text{we accept this critical point since it is on the interval } 0 \leq x \leq 4000$$

$$\hookrightarrow \frac{dP}{dx} \text{ DNE} \Rightarrow \frac{49 - \sqrt{x}}{\sqrt{x}} = \text{DNE}$$

$$\Rightarrow \sqrt{x} = 0$$

$$\Rightarrow x = 0 \leftarrow \text{this is also accepted since it is on the interval } 0 \leq x \leq 4000$$

- ② Calculate function values at the critical points and the end points.

x	$P(x)$
0	$-\$30 \leftarrow P(0) = 98\sqrt{0} - 30 - 0$
2401	$\$2371 \leftarrow P(2401) = 98\sqrt{2401} - 30 - 2401$
4000	$\$2168.06 \leftarrow P(4000) = 98\sqrt{4000} - 30 - 4000$

- ③ Compare values

The absolute projected value is \$2371 if \$2401 is spent on advertising.

- ④ when $x = \$2401$, we expect to sell $q = 0.2\sqrt{2401} = 9.8 \approx 10$ appliances.

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Graphing Functions

- i) State the domain of $f(x)$
- ii) Find any x - and y -intercepts of the graph
- iii) Find any vertical and horizontal asymptotes, and state what the graph is doing as x goes to $-\infty$ and ∞ , if applicable.
- iv) Find the intervals where the graph increases and decreases, and locate any relative extreme values.
- v) Find the intervals where the graph is concave up and concave down, and locate any inflection points.

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$f(x) = \frac{x^2}{x^2+1}$ i) This is a rational function. The domain is everywhere except where the denominator $= 0$.

So we need $x^2+1 \neq 0$. In this example, we see that $x^2+1 \neq 0$ for all x . So, the domain for $f(x) = \frac{x^2}{x^2+1} = -\infty < x < \infty$.

ii) To find the y -intercept, assign $x=0$ and calculate y .

So, $y = \frac{0^2}{0^2+1} = 0$, x -intercept occurs at $(0, 0)$

To find x -intercepts, assign $y=0$ and solve for x .

So, $0 = \frac{x^2}{x^2+1} \Rightarrow x^2 = 0 \Rightarrow x = 0$, y -intercept occurs at $(0, 0)$

iii) Find asymptotes. $y = \frac{x^2}{x^2+1}$ is rational, so we can perform a division since the denominator is non-linear so it is better to perform long division.

$$\text{So, } \frac{x^2}{x^2+1} = \frac{x^2+0x+0}{x^2+0x+1}$$

$$\begin{array}{r} x^2 + 0x + 1 \overline{) x^2 + 0x + 0} \\ \underline{x^2 + 0x + 1} \\ -1 \end{array}$$

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$$y = \frac{x^2}{x^2+1}$$

Recall: $y = \frac{g(x)}{h(x)} \leftarrow$ irreducible

$$\Rightarrow y = 1 - \frac{1}{x^2+1}$$

$$\Rightarrow y = S(x) + \frac{E(x)}{h(x)} \leftarrow \text{proper rational}$$

$\uparrow$ polynomial

Non vertical asymptote

$$is y=1$$

we have i) non-vertical asymptote $y=S(x)$
ii) vertical asymptotes are the solutions to $h(x)=0$

vertical asymptotes are the solutions to

$$x^2+1=0$$

$\Rightarrow x^2 = -1$, which has no solution.

So, $y = \frac{x^2}{x^2+1}$ has no vertical asymptotes.

iv) Find intercepts of increase and decrease and classify the relative extreme points, if any.

$$\text{Given } y = \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = 0 - \left(-\frac{(x^2+1)'}{(x^2+1)^2} \right)$$

Recall: reciprocal rule

$$y = \frac{1}{g(x)} \Rightarrow \frac{dy}{dx} = -\frac{g'(x)}{[g(x)]^2}$$

So we have

$\frac{dy}{dx} = \frac{2x}{(x^2+1)^2}$ critical points occur when (i) $\frac{dy}{dx} = 0$, so

$\Rightarrow \frac{2x}{(x^2+1)^2} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$, we accept this as a critical point since it is on the domain.

ii) $\frac{dy}{dx} = \text{DNE} \Rightarrow \frac{2x}{(x^2+1)^2} = \text{DNE} \Rightarrow (x^2+1)^2 = 0 \Rightarrow x^2+1=0 \Rightarrow x^2=-1$

Which has no solution, so $\frac{dy}{dx} = \text{DNE}$ and yields no critical point for this function.

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Sign Analysis for $\frac{dy}{dx} = \frac{2x}{(x^2+1)^2}$

	$x < 0$	0	$x > 0$
Sign of z	positive		positive
Sign of x	negative		positive
Sign of $(x^2+1)^2$	positive		positive
Sign of $\frac{dy}{dx} = \frac{2x}{(x^2+1)^2}$	negative		positive
orientation of $y = \frac{x^2}{x^2+1}$	declining	rel. min	inclining

Relative minimum at $x=0$, function value at $x=0$

$$y = \frac{0^2}{0^2+1} = 0 \therefore \text{relative minimum point is } (0,0).$$

v) Find intervals of concavity and identify any inflection point. We need $\frac{d^2y}{dx^2} = f''(x)$

$$\text{From } \frac{dy}{dx} = \frac{2x}{(x^2+1)^2} \\ \Rightarrow \frac{d^2y}{dx^2} = \frac{(x^2+1)^2(2x)' - (2x)(x^2+1)^2}{((x^2+1)^2)^2}$$

$$= \frac{(x^2+1)^2(2) - (2x)(2(x^2+1)(2x))}{(x^2+1)^4}$$

$$= \frac{2(x^2+1)[x^2+1-4x^2]}{(x^2+1)^4}$$

$$= \frac{2(1-3x^2)}{(x^2+1)^3}$$

Inflection points may occur when i) $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{2(1-3x^2)}{(x^2+1)^3} = 0$$

$$\Rightarrow 2(1-3x^2) = 0$$

$$\Rightarrow 1-3x^2 = 0$$

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$\Rightarrow x^2 = \frac{1}{3} \Rightarrow x = -\frac{1}{\sqrt{3}}$ or $x = \frac{1}{\sqrt{3}} \leftarrow$ we accept these as possible inflection points as they are on the domain.

(ii) $\frac{d^2y}{dx^2} = DNE \Rightarrow \frac{2(1-3x^2)}{(x^2+1)^3} = DNE$

$\Rightarrow (x^2+1)^3 = 0$

$\Rightarrow x^2 + 1 = 0 \Rightarrow x^2 = -1$, which has no solution and yields no possible inflection points for this interval

	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$x > \frac{1}{\sqrt{3}}$	x
Sign of z	+		+		+	
Sign of $1-3x^2$	-		+		-	
Sign of $(x^2+1)^3$	+		+		+	
Sign of $\frac{d^2y}{dx^2} = \frac{2(1-3x^2)}{(x^2+1)^3}$	-		+		-	
Concavity of $y = \frac{x^2}{x^2+1}$	$\cap$		$\cup$		$\cap$	
		inf pt		inf pt		

Function values at the inflection points At $x = -\frac{1}{\sqrt{3}}$

$\therefore y = \frac{\left(-\frac{1}{\sqrt{3}}\right)^2}{\left(-\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{\frac{1}{3}}{\frac{1}{3} + 1}$

$= \frac{1/3}{4/3} = \frac{1}{4} \therefore$ Inflection point $\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

At $x = \frac{1}{\sqrt{3}}, y = \frac{\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{\frac{1}{3}}{\frac{1}{3} + 1} = \frac{1}{4}$

Another inflection point at $\left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

